Final Exam: Wednesday, 12/17

3-6, this room

Anthony will conduct a review session (Tuesday 3:30-5:30 - MLC)

Jordan Canonical Form, Nilpotents

If $A \in M_n(\mathbb{C})$ is nilpotent, then A is similar to a block diagonal matrix, all of whose blocks are of the form

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \text{\Large 0} \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

($[0]$ in the case of a 1×1 block)

The idea: Suppose $A \in M_n(\mathbb{C})$ is nilpotent of index n. We know from a lemma proved in last class that

$$\{A^i x\}_{i=0}^{n-1} \text{ is a}$$

basis for $\mathbb{C}^n$ for some $x \in \mathbb{C}^n$.

Let $y_i = A^{n-i}x$,

$1 \leq i \leq n$.

$$\boxed{\begin{aligned} Ay_i &= A(A^{n-i}x) \\ &= A^{n-i+1}x \\ &= y_{i-1} \end{aligned}}$$

$\forall\ 2 \leq i \leq n$ and

$$\boxed{\begin{aligned} Ay_1 &= A(A^{n-1}x) \\ &= A^n x = 0. \end{aligned}}$$

$$S = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix} \quad (n \times n)$$

Then

$$\boxed{\begin{array}{l} Be_1 = 0 \\ Be_i = e_{i-1} \end{array}} \quad \begin{array}{l} \text{and} \\ \text{for } 2 \le i \le n \end{array}$$

where $\{e_i\}_{i=1}^{n}$ is the standard basis.

Then

$$A = SBS^{-1}.$$

$$SBS^{-1} y_i$$
$$= SBe_i$$
$$= S \begin{cases} 0, & i=1 \\ e_{i-1}, & i>1 \end{cases}$$
$$= \begin{cases} 0, & i=1 \\ y_{i-1}, & i>1 \end{cases}$$
$$= Ay_i$$

Lemma: (basis) Suppose $A \in M_n(\mathbb{C})$ is nilpotent. Then $\exists\ k \in \mathbb{N}$, $n_1, n_2, \dots, n_k \in \mathbb{N}$, and $x_1, x_2, \dots, x_k \in \mathbb{C}^n$ such that

$$\mathcal{B} = \{x_1, Ax_1, \dots, A^{n_1-1}x_1,$$
$$x_2, Ax_2, \dots, A^{n_2-1}x_2,$$
$$\vdots$$
$$x_k, Ax_k, \dots, A^{n_k-1}x_k\}$$

is a basis for $\mathbb{C}^n$.

Moreover, $A^{n_i} x_i = 0$

$\forall\ 1 \le i \le k$.

Proof: Use induction.

$n=1$: 0 is the only nilpotent, nothing to prove.

$n=2$: If $A=0$, nothing to prove.

Suppose $A \neq 0$.

Then $\ker(A)$ is one-dimensional

so $\exists\ x \in (\ker(A))^{\perp},\ x \neq 0.$

Moreover, A has index 2,

so $\quad Ax \neq 0 \quad (x \notin \ker(A))$

but $A^2x = 0x = 0.$

By the lemma from last class, $\{x, Ax\}$ is a basis for $\mathbb{C}^2$.

$\underline{n=3}$ $A=0$, trivial.

So suppose $A \neq 0$.

The index of A is either 2 or 3. If the index of A is 3:

$\exists\ x \in \mathbb{C}^3$,

$$A^2x \neq 0 \text{ but } A^3x = 0.$$

Hence by the lemma,

$\{x, Ax, A^2x\}$ is a basis for $\mathbb{C}^3$.

If the index is 2:

Consider $\operatorname{ran}(A)$.

$A \neq 0$ and nilpotent

$\Rightarrow 0 < \operatorname{rank}(A) < 3$.

$\boxed{\text{Suppose } \operatorname{rank}(A) = 2.}$

Then with $V = \operatorname{ran}(A)$,

$A : V \to V$.

Apply induction to obtain an $x \in V$ such that $\{x, Ax\}$ is a basis for V. But $V = \operatorname{ran}(A)$, so $\exists\ y \in \mathbb{C}^3$,

$$x = Ay.$$

Then $Ax = A^2 y$

$$= 0$$

contradiction.

Therefore, $\operatorname{rank}(A) = 1$.

By rank-nullity,

$$\begin{aligned} \dim(\ker(A)) &= 3-1 \\ &= 2. \end{aligned}$$

$$A : \ker(A) \to \ker(A)$$

Let $\{x_1, x_2\}$ be a basis for $\ker(A)$.

Choose $y \in \operatorname{Ran}(A)$,

$y = Ax_3$ for some $x_3 \in \mathbb{C}^3$.

Claim: $\{x_1, x_2, x_3\}$ is a basis for $\mathbb{C}^3$.

Suppose $\exists\ \alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0.$$

Apply A

$$0 = A0 = A(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)$$

$$= \underbrace{\alpha_1 A x + \alpha_2 A x_2}_{0 \text{ since } x_1, x_2 \in \mathrm{Ker}(A)} + \alpha_3 A x_3$$

$$= \alpha_3 A x_3 = \alpha_3 y.$$

We can choose y nonzero, forcing $\alpha_3 = 0$.

Then $\alpha_1 x_1 + \alpha_2 x_2 = 0$

But $\{x_1, x_2\}$ is a basis for $\ker(A)$, so $\alpha_1 = \alpha_2 = 0$.

Therefore $\{x_1, x_2, x_3\}$ is the desired basis.

<u>General case</u>: suppose

$n > 3$.

Consider $V = \mathrm{ran}(A)$,

$\mathrm{rank}(A) = \dim(\mathrm{ran}(A)) < n$.

If $A \neq 0$, $\mathrm{rank}(A) > 1$.

Since $A : V \to V$, we apply induction on V.

$\exists\ m \in \mathbb{N},\ n_1, n_2, \ldots, n_m \in \mathbb{N}$

$y_1, y_2, \ldots, y_m \in V$

$$\mathcal{B}_0 = \{ y_1, Ay_1, \ldots, A^{n_1-1} y_1,$$
$$y_2, Ay_2, \ldots, A^{n_2-1} y_2,$$
$$\vdots$$
$$y_k, Ay_k, \ldots, A^{n_m-1} y_k \}$$

is a basis for V and

$A^{n_i} y_i = 0 \quad \forall\ 1 \le i \le m$

$\Rightarrow A^{n_i - 1} y_i \in \mathrm{Ker}(A)\ \forall\ 1 \le i \le m$

$\exists\ x_1, x_2, \ldots, x_m \in \mathbb{C}^n,$

$y_i = Ax_i \quad \forall\, 1 \leq i \leq m$

$(\mathcal{V} = \mathrm{ran}(A)).$

Claim: $\mathcal{B}_1 = \mathcal{B}_0 \cup \{x_i\}_{i=1}^{m}$

is linearly independent

Suppose $\exists$ scalars

$\{\alpha_{i,j}\}_{i=1,\, j=1}^{m\quad n_j}$ and

$\{\beta_i\}_{i=1}^{m}$ with

$$\sum_{j=1}^{m}\sum_{i=1}^{n_j}\alpha_{i,j}A^{n_j-i}y_j$$

$$+\sum_{j=1}^{m}\beta_j x_j = 0.$$

Apply A

$$0 = A(0)$$

$$= A\left(\sum_{j=1}^{m}\sum_{i=1}^{n_j}\alpha_{i,j}A^{n_j-i}y_j\right)$$

$$+ A\left(\sum_{j=1}^{m}\beta_j x_j\right)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n_j} \alpha_{i,j} A^{n_j - i + 1} y_j$$

$$+ \sum_{j=1}^{m} \beta_j A x_j$$

$$= \sum_{j=1}^{m} \sum_{i=2}^{n_j} \alpha_{i,j} A^{n_j - i + 1} y_j \quad \text{since } A^{n_j} y_j = 0$$

$$+ \sum_{j=1}^{m} \beta_j y_j$$

$$\Rightarrow \quad \beta_j = 0 \ \forall \ 1 \le j \le m \text{ and}$$

$$\alpha_{i,j} = 0 \ \forall \ 1 \le j \le m, \ 2 \le i \le n_j$$

by linear independence of $\mathcal{B}_0$.

Going back to the original sum, we have

$$\sum_{j=1}^{m} \alpha_{1,j} A^{n_j - 1} y_j = 0$$

$\Rightarrow \alpha_{1,j} = 0 \ \forall \ 1 \leq j \leq m$,

again by linear independence of $\mathcal{B}_0$

Hence, $\mathcal{B}_1$ is linearly independent.

Recall $\{A^{n_i-1}y_i\}_{i=1}^{m} \subseteq \ker(A)$

and is linearly independent

since it is a subset of $\mathcal{B}_0$

Extend $\{A^{n_i-1}y_i\}_{i=1}^{m}$ to

a basis of $\ker(A)$ by adding

vectors $\{z_i\}_{i=1}^{r}$.

Let $\mathcal{B} = \mathcal{B}_1 \cup \{z_i\}_{i=1}^{r}$.

Claim: β is linearly independent

Suppose $\exists$ scalars

$$\left\{\left\{\alpha_{i,j}\right\}_{i=0}^{n_j}\right\}_{j=1}^{m} \text{ and}$$

$$\left\{\gamma_i\right\}_{i=1}^{r} \text{ with}$$

$$\sum_{j=1}^{m}\sum_{i=0}^{n_j}\alpha_{i,j}A^{n_j-i}x_j$$

$$+\sum_{i=1}^{r}\gamma_i z_i = 0$$

Apply A.

$$0 = A(0)$$

$$= A\left(\sum_{j=1}^{m}\sum_{i=0}^{n_j}\alpha_{i,j}A^{n_j-i}x_j + \sum_{i=1}^{r}\gamma_i z_i\right)$$

$$= \sum_{j=1}^{m}\sum_{i=0}^{n_j}\alpha_{i,j}A^{n_j-i+1}x_j + \underbrace{\sum_{i=1}^{r}\gamma_i A z_i}_{=0 \text{ since } \{z_i\}_{i=1}^{r} \subseteq \ker(A)}$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n_j}\alpha_{i,j}A^{n_j-i+1}x_j$$

Since $A^{n_j+1}x_j = A^{n_j}y_j = 0 \;\forall\, 1 \le j \le m$

$\Rightarrow \quad \alpha_{i,j} = 0 \quad \forall \quad 1 \le j \le m,$
$\quad 1 \le i \le n_j$

by linear independence of $\mathcal{B}_1$.

Going back to the original sum,

$$0 = \sum_{j=1}^{m} \alpha_{0,j} A^{n_j} x_j + \sum_{i=1}^{r} \gamma_i z_i$$

$\Rightarrow \quad \gamma_i = 0 \quad \forall \quad 1 \le i \le r$

and $\alpha_{0,j} = 0 \quad \forall \quad 1 \le j \le m$

Since $\{A^{n_j}x_j\}_{j=1}^{m} \cup \{z_i\}_{i=1}^{r}$ is a basis for $\ker(A)$.

Hence, $\mathcal{B}$ is linearly independent.

Set $x_{m+i} = z_i$, $1 \le i \le r$.

Note $\left| \{A^j x_i\}_{i=1}^{m} {}_{j=0}^{n_i - 1} \right|$

$= \left| \{A^j x_i\}_{i=1}^{n} {}_{j=1}^{n_i} \right|$

$= |\mathcal{B}_0| = \operatorname{rank}(A)$.

By rank-nullity,

$$\dim(\ker(A)) = n - \operatorname{rank}(A),$$

hence

$$|\mathcal{B}|$$

$$= \left| \{A^i x_j\}_{j=1}^{m}{}_{i=0}^{n_j-1} \cup \{A^{n_j} x_j\}_{j=1}^{m} \cup \{x_i\}_{i>m} \right|$$

$$= \left| \{A^i x_j\}_{j=1}^{m}{}_{i=0}^{n_j-1} \right| + \left| \{A^{n_j} x_j\}_{j=1}^{m} \cup \{x_i\}_{i>m} \right|$$

$$= |\mathcal{B}_0| + \dim(\ker(A))$$

$$= \operatorname{rank}(A) + (n - \operatorname{rank}(A)) = n$$

Then $\mathcal{B}$ is the desired basis. $\square$

Jordan Canonical Form for Nilpotents

Let $A \in M_n(\mathbb{C})$ be nilpotent.

Use the lemma to find a basis

$$\mathcal{B} = \{A^{n_j - i} x_j\}_{j=1}^{k} {}_{i=1}^{n_k}.$$

Let $y_{i,j} = A^{n_j - i} x_j$.

Let

$$S = \begin{bmatrix} \\ y_{1,1} \cdots y_{n_1,1} & y_{2,1} \cdots y_{2,n_2} & \cdots & y_{k,1} & \cdots & y_{k,n_k} \\ \\ \end{bmatrix}$$

Then

$$S^{-1}AS = \begin{array}{cc} & \begin{array}{cccc} n_1 & n_2 & \cdots & n_k \end{array} \\ \begin{array}{c} n_1 \\ \\ n_2 \\ \vdots \\ n_k \end{array} & \begin{bmatrix} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & 0 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & 0 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & 0 & 0 \end{matrix} \end{bmatrix} \end{array}$$